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# Application of the Generalized Short Time DFT to the Hilbert Transformer and Its Characteristics

# Generalized Short Time DFTのヒルベルト変換器への適用とその特性

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*ABSTRACT* *Instantaneous spectrum concept is a promising solution to effectively developing key devices for economical communication systems. An exact realization of the Hilbert transformer has been previously discussed with employing new concept of instantaneous spectrum defined by ST DFT. The Hilbert transformer used in SSB or RZ SSB modulator provides with indispensable function for eliminating one sideband from output signals to efficiently reduce occupied spectrum over radio channels. A new class of signal processing is introduced by generalized short time DFT, in which sub-channels are arbitrary adjusted on the objective frequency domain. Another implementation of the noble Hilbert transformer is discussed with employing this ST gDFT. Restricting the ST gDFT within causality, phase shifting error of implemented Hilbert transformer is examined to be so accurate as detecting no error by micro degree order. Simultaneously, its amplitude error is shown to be less than 0.01dB.*

## 1. INTRODUCTION

Narrowing frequency bandwidth occupied over radio channel is a promising solution to effective usage of finite radio resource. SSB has been considered to be defective in vehicular communication systems via such speech quality degradation as fading on the multi-pass propagation, although SSB is the most efficient in narrowing frequency bandwidth occupancy of modulated signals. However, RZ SSB which has equal frequency utilization efficiency to the existing SSB has been pushed on the stage of developing new modulation technique which guarantees equal speech quality to the existing PM modulation over fading poor radio channels.[1, 2] As known well, the noble Hilbert transformer plays the important value of making the frequency utilization efficiency so high with generating analytic signals through modulation.[3-5] Hilbert transformed signal $\hat{f}(t)$ is exactly generating from inverse Fourier transform of $\{-j\,sign(\omega)F(\omega)\}$, here $F(\omega)$ is the Fourier transform of $f(t)$. This relationship has suggested previously that Hilbert transformer is executed on the phase plane. Once the instantaneous spectrum are given, Hilbert transformer is precisely carried out on the phase plane via merely exchanging real or imaginary part of these spectrum with each other.[6] Hilbert transformed signals are, therefore, synthesized from these exchanged spectrums as shown in session 3. A new class of analysis and synthesis named by generalized ST gDFT is proposed with emphasis on generalizing the sub-channel allocation on the

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phase plane. ST gDFT promises a precise signal processing for implementation of noble Hilbert transformer because of the characteristics, evenness of real function $R(\omega)=R(-\omega)$ and oddness of imaginary function $X(\omega)=-X(\omega)$. Noble Hilbert transformer, which is called by ST gDFT Hilbert transformer, is newly proposed in this paper on the newly proposing signal processing concept of the ST gDFT.

Circuit configuration of this ST gDFT Hilbert transformer is discussed in brief and verified its characteristics through computer simulations.

## 2. HILBERT TRANSFORMER

A real signal $\hat{f}(t)$ is defined at almost all t by inverse Fourier transform from Fourier transform $F(\omega)$ of arbitrary signal $f(t)$ whose real or imaginary part is exchanged with each other. This real signal $\hat{f}(t)$ is Hilbert transform of $f(t)$ That is,

$$\hat{f}(t) = \frac{1}{\pi}\int_0^\infty \{R(\omega)\sin\omega t + X(\omega)\cos\omega t\}\, d\omega. \tag{1}$$

Where the real part $R(\omega)$ is even function given as

$$R(\omega)=\frac{1}{2}\{F(\omega)+F(-\omega)\},$$

the imaginary $X(\omega)$ is odd function given as

$$X(\omega)=\frac{1}{2j}\{F(\omega)-F(-\omega)\},$$

$\hat{f}(t)$ is therefore given as follows.

$$\begin{aligned}\hat{f}(t) &= \frac{1}{\pi}\int_0^\infty\Big[\frac{1}{4j}\{F(\omega)+F(-\omega)\}\{e^{j\omega t}-e^{-j\omega t}\} + \frac{1}{4j}\{F(\omega)-F(-\omega)\}\{e^{j\omega t}+e^{-j\omega t}\}\Big]\, d\omega\\ &= \frac{1}{2\pi}\int_0^\infty\{-jF(\omega)e^{j\omega t} + jF(-\omega)e^{-j\omega t}\}\, d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty -j\,sign(\omega)\, F(\omega)\, e^{j\omega t}\, d\omega.\end{aligned}$$

The Hilbert transform on the phase plane is described as

$$\hat{F}(\omega) = -j\,sign(\omega)\,F(\omega) \tag{2}$$

On the phase plane, Hilbert transform is interpreted as filtering by $-j\,sign(\omega)$. Equation 2 shows that the ideal Hilbert transform is obtained from shifting the phase by $-90°$ $(\omega>0)$ and by $90°$ $(\omega<0)$ during signal processing based on the instantaneous spectrum of the ST gDFT.

Figure 1 shows frequency response of the Hilbert transform as filtering in the discrete signal processing system. The unit sample response $i(n)$ of this system is given by

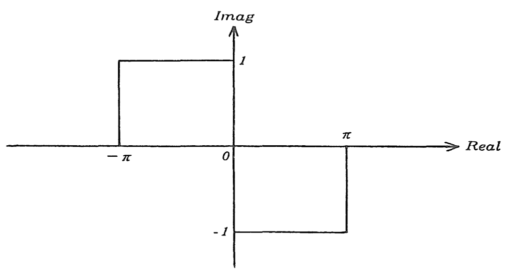


*Fig.1 Frequency Response of the ideal Hilbert transform in the discrete signal processing systems.*

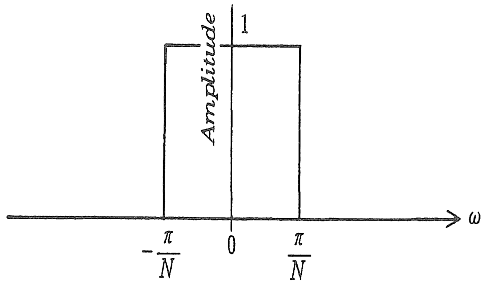


Fig.2 The $0_{th}$ sub-channel allocation of the existing DFT

$$
\begin{aligned}
i(n) &= \frac{1}{N}\left\{\int_{-\frac{N}{2}}^{0} je^{j\frac{2\pi}{N}\omega n}\right.\\
&\quad + \left.\int_{0}^{\frac{N}{2}} -je^{j\frac{2\pi}{N}\omega n}\right\}dw \\
&= \frac{1}{2\pi n}\{(1-e^{-j\pi n})-(e^{j\pi n}-1)\} \\
&= \frac{1}{\pi n}(1-\cos\pi n)
\end{aligned} \tag{3}
$$

This unit sample response $i(n)$ is exactly same to that of Rabiner's minmax Hilbert transformer $i_m(n)$.[7,8]

$$
i_m(n) = \begin{cases} \dfrac{2}{\pi}\dfrac{\sin^2(\frac{\pi n}{2})}{n}, & if \quad n \quad is \quad odd \\ 0, & if \quad n \quad is \quad even \end{cases} \tag{4}
$$

The existing DFT is impossible to be applied to the Hilbert transformer immediately because of $0th$ sub-channel existing on the frequency domain $(-\frac{\pi}{N}, \frac{\pi}{N})$ accrossing zero as shown in fig.2.

New signal processing is discussed in the followings to solve this problem of $0th$ sub-channel merely by shifting channel allocation by half along to the frequency axis based on generalized short Time DFT, which is also newly proposing here to coincide the fringe of $0th$ sub-channel to zero.

## 3. GENERALIZED SHORT TIME HILBERT TRANSFORMER

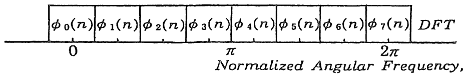


*Fig.3 Comparison of sub-chanel allocation between existing DFT and ST gDFT, N=8*

### 3.1 Definition of the ST gDFT and Hilbert Transformer Operator

We define ST gDFT and ST gIFT as follows.[9]

*ST gDFT* :

$$
\phi_k(n) = \sum_{r=-\infty}^{\infty} x(r)h(n-r)W_N^{-(k+\xi)r} \tag{5}
$$

*ST gIFT* :

$$
y(n) = \frac{1}{N}\sum_{k=0}^{N-1} \phi_k(n)W_N^{n(k+\xi)} \tag{6}
$$

Here, $\xi$ is positive real number, $0 \leqq \xi < 1$.

$x(r)$ is an input data at sampling time $r$.

$h(n-r)$ is the same window functions as define in ST DFT,

$$
h(p) = \begin{cases} 1, & if \ p=0 \\ 0, & if \ p=Nu, \end{cases} \tag{7}
$$

$u$ *is non zero integer*

This satisfies physical existence and stands on causality to exist complex conjugate structure with symmetric axis at $\pi$ radian among spectrum components, if $\xi$ is $\frac{1}{2}$, as shown in fig. 3.

Hilbert transform is executed by exchanging complex components of $x(n)$ with each other on the frequency domain as discussed in the previous session.

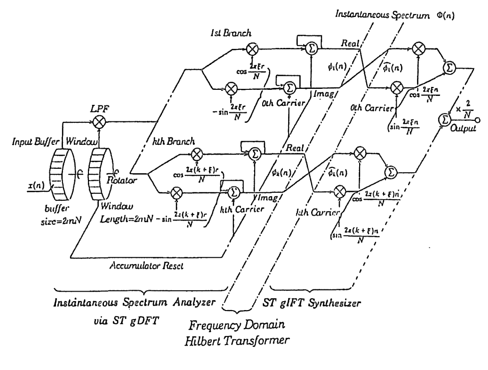


*Fig.4 Circuitry configuration of ST gDFT Hilbert transformers*

Let complex real and imaginary part of $\phi_k(n)$ to be $a_k(n)$ and $b_k(n)$, respectively. The transformed components $\widehat{\phi}_k(n)$ are given as follows,

$$\widehat{\phi}_k(n) = b_k(n) - ja_k(n) = -j\{a_k(n) + jb_k(n)\} \tag{8}$$

As shown in eq.8, vector $\widehat{\phi}_k(n)$, which is spanned by mutually exchanged complex components, is precisely coincident with the $k_{th}$ component of Hilbert transformed instantaneous spectrum.

Both instantaneous spectrum analysis and phase shifting being combined into one operator, the frequency domain Hilbert transform operator $\widehat{W}_N^{-rk}$ is given as follows,

$$\widehat{W}_N^{-rk} = \begin{cases} exp\left\{-j\frac{2\pi}{N}(k+\xi)r - j\frac{1}{2}\right\}, \\ \quad if\ 0 \leqq k < \frac{N}{2} \\ 0, \\ \quad if\ k = \frac{N-1}{2} \\ \quad N\ is\ odd \\ exp\left\{-j\frac{2\pi}{N}(k+\xi)r + j\frac{\pi}{2}\right\}, \\ \quad if\ \frac{N}{2} \leqq k < N \end{cases} \tag{9}$$

Substituting $\widehat{\phi}_k(n)$ into eq.6, ST gIFT gives the corresponding Hilbert transformed signal $\widehat{y}(n)$ as follows.

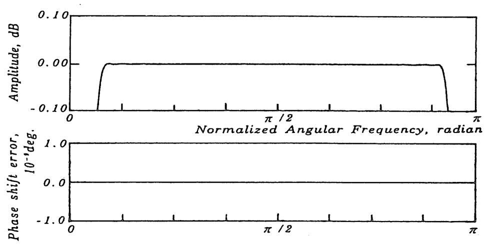


*Fig.5 Frequency response of amplitude and phase shift error of the ST gDFT Hilberttransformer, 2m=8 and β=9.0*

$$\begin{aligned} \widehat{y}(n) &= \frac{1}{N}\sum_{k=0}^{\frac{N}{2}-1}\left\{\widehat{\phi}_k(n)W_N^{n(k+\xi)} \right. \\ &\quad \left. \widehat{\phi}_{N-1-k}(n)W_N^{-n(k+1-\xi)}\right\} \\ &= \frac{1}{N}\sum_{k=0}^{\frac{N}{2}-1}\left\{\widehat{\phi}_k(n)W_N^{n(k+\xi)} \right. \\ &\quad \left. + \overline{\widehat{\phi}_k(n)}\overline{W_N^{n(k+\xi)}}\right\} \\ &= \frac{2}{N}\sum_{k=0}^{\frac{N}{2}-1} Real\left\{\widehat{\phi}_k(n)W_N^{n(k+\xi)}\right\}, \qquad QED \\ &\quad here\ \ \xi = \frac{1}{2}\ . \end{aligned} \tag{10}$$

**3.2 Circuitry Configuration and Its Unit Sample Response**

The ST gDFT Hilbert transformer consists of three major blocks as shown in fig.4. The first plays a role of ST gDFT analyzer and consists of $\frac{N}{2}$ modules in which every component $\phi_k(n)$ of the instantaneous spectrum $\Phi(n)$ is yielded. The second block acts as a Hilbert transformer on the frequency domain, which exchanges real with imaginary part of $\phi_k(n)$ . This block is dominant in function, however, it is so simple in circuitry configuration as merely consisting of two crossing wires as shown in fig.4. These blocks are practically combined together in frequency index wise to get $\widehat{\phi}_k(n)$ directly by adopting $\widehat{W}_N$ instead of $W_N$ during ST gDFT analyzing. The last is synthesiz-

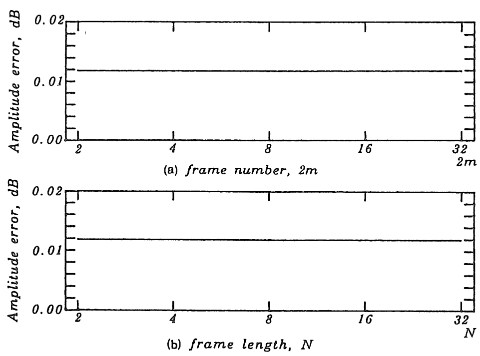


*Fig.6 Amplitude error vs. frame number 2m (a),and amplitude error vs. frame length N (b) of the ST gDFT Hilbert transformer, 2mN =64 and β=6.0*

er which employs ST gIFT to produce time from Hilbert transformed spectrum $\widehat{\Phi}(n)$ in similar to the first blocks. The unit sample response $i_g(n)$ of the ST gDFT Hilbert transformer is given by eq.11.

$$i_g(n) = \begin{cases} \frac{1}{N} h(n) \frac{2}{\sin\frac{n\pi}{N}}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases} \tag{11}$$

It is easy to understand that $i_g(n)$ converges onto unit sample response of the ideal Hilbert transform if $h(n)$ is an infinite frame number Nyquist. In fact, $h(n) = \sin\frac{n\pi}{N} / \frac{n\pi}{N}$ being substituted into eq.11, $i_g(n)$ gives ideal response as follows.

$$i_g(n) = \begin{cases} \frac{2}{n\pi}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases} \tag{11'}$$

Attention must be paid on that the frame length $N$ does not effect the unit sample response, where the ideal Hilbert transformer response is defined by that of ST gDFT Hilbert transformer as shown eq.11'. If the infinite Nyquist window is used, the ideal Hilbert transform is easy to realize but be fatal in implementation owing to output signal being delayed by infinite duration. Fortunately, ST gDFT Hilbert transform is defined with processing input signals on the frequency domain.

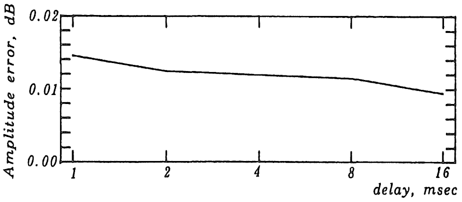


*Fig. 7 Amplitude error vs. delay mN of the ST gDFT Hilbert transformer, β =6.0 and sampling rate is 8kHz*

Therefore, it becomes to possible in the ST gDFT to employ finite frame number $h(n)$ in order to shift the phase by 90° exactly.

Consider the Kaiser smoothing function to truncate infinite Nyquist by finite frame number,[10]

$$h(n) = N(n)K(\beta, n) \tag{12}$$

here, $N(n)$ is infinite Nyquist

$$N(n) = \frac{\sin\frac{n\pi}{N}}{\frac{n\pi}{N}},$$

and $K(\beta, n)$ is truncating function

$$K(\beta, n) = \frac{I_0\left(\beta\sqrt{1-\frac{n^2}{m^2N^2}}\right)}{I_0(\beta)}, \quad -mN \leqq n \leqq mN. \tag{13}$$

Where, $I_0(*)$ is the modified $0_{th}$ order first kind Bessel function, $\beta$ is arbitrary value to adjust width and energy of the mainlobe.

It will be shown in the next session that the truncated window $h(*)$ is approximately adjusted to coincide with the infinite Nyquist with selecting $\beta$ by apriori values. Function $N(n)K(\beta, n)$ is especially called by Nyquist-Kaiser in the followings.

## 4. RESULT OF COMPUTER SIMULATIONS

The amplitude frequency response is shown in fig.5 for unit sample response of the ST gDFT Hilbert transformer in which the frame number $2m$ is set to be 8 and $\beta$ of Nyquist-Kaiser is taken as 9. As shown in flatness over subjective domain, the optimized approximation is easily given by adjusting the value of $\beta$. It is also mentioned that there exists any phase shift error in accuracy of micro degree order.

Figure 6 (a) shows the amplitude response of the ST gDFT Hilbert transformer as the frame number $2m$ being taken as a parameter when the Nyquist-Kaiser window length $2mN$ is set to be 64 and $\beta$ is 6. Under the same conditions in the above, figure 6 (b) shows the amplitude response as the frame length $N$ being taken as a parameter. As shown in these figures, the Hilbert transformer is low in sensitivity to cause no changes in amplitude characteristics if the parameter $2m$ or $N$ changes.

Even if it gives good characteristics when infinite Nyquist being employed, it is not practicable because of being large in processing delay. It is easy to understand that only the single function of the Hilbert transformer is also realized with transversal filters when the unit sample response $i_g(n)$ of the ST gDFT Hilbert transformer is exactly given. This means the delay of ST gDFT Hilbert transformer is given by $mN\tau$. Here, $\tau$ is reciprocal number of sampling frequency.

Figure 7 shows the amplitude response as delay $mN$ being taken as a parameter. The amplitude error of the ST gDFT Hilbert transformer is shown to be practicable from the value observed in fig.7 to be below 0.01dB when its processing delay is restricted within 16 msec in the case of 8 kHz sampling as standard in communication signal processing.

## 5. CONCLUSION

The generalized short time DFT abbreviated as ST gDFT was successfully shown to be deduced from adjusting allocation of sub-channels with emphasis on realization of the noble Hilbert transformer which is inevitable in improving frequency utility efficient of communication systems. The unit sample response of the ST gDFT Hilbert transformer which employs the infinite Nyquist window is precisely coincide with that of ideal Hilbert transformer with fatal demerit of astronomical delay. However, the ST gDFT Hilbert transformer is executed with signal processing on the phase plane through instantaneous spectrum analysis and synthesis based on the ST gDFT to avoid this fatal demerit and to be able to get exactly Hilbert transformed signal within practicable delay, $mN\tau$.

The Kaiser function introduced into the ST gDFT is also able to speed up the signal processing by truncating Nyquist function without increasing both phase shifting and amplitude errors. The ST gDFT Hilbert transformer is shown to be released from the restrict conditions of frame number $2m$ and frame length $N$ and shown to be depend on only the product $2m \times N$ which is proportional to the delay amount. The frequency responses are verified through computer simulations to be so accurate as less than micro degree order in phase shifting and less than 0.01 dB in amplitude error.

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